Geometric interpretation of spinor and gauge vector

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(Dated: February 1, 2008)

In the standard model, electroweak theory is based on chiral gauge symmetry, which only contains massless fermions and gauge vectors. So the fundamental field equations are Weyl equations including gauge vectors. We deduced these Weyl equations without gauge vector from quadric surface in three dimensional oriented projective geometry. It showed that Weyl equations were equations of generating lines in the quadric surface. In other word, the geometric correspondence of spinor is generating lines of quadric surface in momentum space. It also showed that two spinors would generate a gauge vector. After this quadric surface was localized, its off-origin generating line would correspond to global Weyl equation including gauge vector, from which we deduced the global and local gauge transformation.

PACS numbers: 04.20.Gz, 11.30.Rd, 11.25.Mj

I. INTRODUCTION

In modern physics, spinor analysis has been a basis in field theory, such as unified field theory, general relativity. It is well known that spinor is more fundamental than vector, because a vector is a dyad of conjugate spinors. Though spinor is so fundamental, we can not visualize it in our mind. The simplest, two-component spinor comes from Weyl equation, i.e. field equation of massless fermions. Then, we expect to be able to find the geometric correspondence of spinors by the geometric interpretation of Weyl equations.

This paper includes five parts: First, Weyl equations are reviewed briefly in II. Secondly, the special relativity are described by a convenient geometric method in III. Thirdly, the equivalence between equations of generating lines in quadric surface and Weyl equations is deduced in IV. Fourthly, the geometric interpretation of spinor representation of gauge vector are given in V. And in VI we deduce the global and local gauge transformation. At the end of paper, we conclude in VII.

We use a system of unit in which $\hbar = c = 1$ in this paper.

II. WEYL EQUATIONS

Historically, Dirac[1] formulated his four-component relativistic wave equation in 1928. Subsequently, Weyl[2] proposed two-component relativistic wave equation to describe the massless fermions in 1929. But it was used as neutrino field equation until 1957[3, 4, 5]. Later, the maximum parity violation of weak interaction made Weyl equation become the fundamental fermion field equations in electroweak theory[6, 7], because this two-component equation satisfies the chiral gauge symmetry of electroweak theory as required.

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Weyl equations have two types, i.e.

$$\partial_t \psi \;=\; \boldsymbol{\sigma} \cdot \nabla \psi \tag{1a}$$

and

$$\partial_t \tilde{\psi} \;=\; -\boldsymbol{\sigma} \cdot \nabla \tilde{\psi}, \tag{1b}$$

where the σ's are 2×2 matrices. One of suitable choice of this matrices is Pauli spin ones:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The wavefunction ψ must have two components, since it is operated on by 2×2 matrices. Thus,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3}$$

Then, Weyl equations satisfy the relativistic relation:

$$E^2 = \boldsymbol{p}^2. \tag{4}$$

In fact, the equation (1a) and (1b) are correlative, because if ψ is a solution of equation (1a), it is easily proved that

$$\tilde{\psi} \equiv \sigma_2 \psi^* \tag{5}$$

is a solution of equation (1b), where ψ^* is complex conjugate of ψ.

Consider the plane-wave solutions of Weyl equation (1a):

$$\psi(\boldsymbol{x}, t) = \psi(\boldsymbol{p}, E) e^{i(\boldsymbol{px} - Et)}. \tag{6}$$

Substituting (6) in (1a) we get

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} \psi(\boldsymbol{p}, E) = -E \psi(\boldsymbol{p}, E), \tag{7}$$

i.e.

$$\begin{pmatrix} p_z + E & p_x - i p_y \\ p_x + i p_y & -p_z + E \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{8}$$

In the same way, we can get the following equation from (1b):

$$\boldsymbol{\sigma} \cdot \boldsymbol{p}\tilde{\psi}(\boldsymbol{p}, E) = E\tilde{\psi}(\boldsymbol{p}, E). \tag{9}$$

After defining helicity operator to be $H = \boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$, it is deduced that in Weyl equation (1a) the eigenvalue of H is always -1 for positive energy and $+1$ for negative one, i.e., positive particles are in left-handness and negative one in right-handness. On the contrary, in (1b) positive particles are in right-handness and negative one in left-handness. On the other hand, we can get Dirac massless equation from one first type and one second type Weyl equation, or from two first type ones because of correlativity (5).

III. GEOMETRY

Now we shall think of the minimal homogeneous (Klein) geometry which Weyl equations could be able to be embedded. We call this geometry the null relativistic geometry for without mass term.

First, we could see that either of Weyl equations have four terms, i.e.,

$$\sigma_1 p_x \psi + \sigma_2 p_y \psi + \sigma_3 p_z \psi \pm E\psi = 0. \tag{10}$$

This requires that there must be at least four coordinates in this homogeneous geometry.

Second, the components of wave function in Weyl equations (10) are all complex functions, then the geometry should be complex. Then the required homogeneous geometry is of four complex coordinates.

Third, according to Klein's Erlangen program [8], objects of geometry are the invariant and invariance under transformation group, then we shall determine the transformation group of this geometry, which includes the symmetry group of Weyl equations as a subgroup.

As the relativistic field equations of fermions, Weyl equations satisfy naturally the requires of relativity. In contrast with special and general relativity in table below, Weyl equations have more symmetry than special relativity. Its symmetry group is conformal group, $SO(4, 2) \simeq SU(2, 2)$. So, this requires that the transformation group of this geometry must include $SU(2, 2)$ as a subgroup.

Relativity	Null	Special	General
Mass term	no	inertial	gravitational ($\cong$ inertial)
Symmetry group	conformal group $SO(4, 2) \simeq SU(2, 2)$	Poincaré group $SO(3, 1) \otimes P(4)$	Lorentz group $SO(3, 1)$
Constraint	$\boldsymbol{p}^2 = E^2$	$\boldsymbol{p}^2 + m^2 = E^2$	$G_{\mu\nu} = -8\pi G T_{\mu\nu}$
Geodesic	$0 = \boldsymbol{x}^2 - t^2$	$ds^2 = \boldsymbol{x}^2 - t^2$	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

All in all, the null relativistic geometry is complex geometry, which has four coordinates, and whose transformation group includes $SU(2, 2)$ as a subgroup. Then the minimal geometry is three dimensional oriented complex projective geometry $O\mathbb{C}P^3 = \mathbb{C}P^3 \otimes Z_2$ [9, 10], whose transformation group is $SL(4, \mathbb{C})$, or four dimensional complex vector space $\mathbb{C}^4$ [11], whose transformation group is $GL(4, \mathbb{C}) = SL(4, \mathbb{C}) \otimes U(1, \mathbb{C})$. The projective geometry is preferred to discuss the geometric mean of Weyl equations. To express a point in three dimensional projective geometry, we need four homogeneous coordinates (x_1, x_2, x_3, x_4), In this way, the ordinary points could be expressible as $(x_1, x_2, x_3, 1)$, and infinity ones as $(x_1, x_2, x_3, 0)$. While considering the geometric expression in four dimensional spacetime, we shall turn to $\mathbb{C}^4$.

IV. GEOMETRIC INTERPRETATION

Now we shall find the geometric interpretations of Weyl equations in $O\mathbb{C}P^3$. The goal is geometric correspondence of Weyl equations.

Because Weyl equations must satisfy the constrain $p_x^2 + p_y^2 + p_z^2 - E^2 = 0$, its geometric correspondence would lie in the quadratic surfaces $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ of $O\mathbb{C}P^3$. Now, we decompose this quadratic surface as below:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 0 \\ \Rightarrow\ (x_1 + ix_2)(x_1 - ix_2) &= -(x_3 + ix_4)(x_3 - ix_4) \end{aligned}$$

so,

$$-\frac{x_1 - ix_2}{x_3 + ix_4} = \frac{x_3 - ix_4}{x_1 + ix_2} = \frac{\psi_1}{\psi_2} = \lambda \tag{11a}$$

$$-\frac{x_1 - ix_2}{x_3 - ix_4} = \frac{x_3 + ix_4}{x_1 + ix_2} = \frac{\tilde{\psi}_1}{\tilde{\psi}_2} = \mu. \tag{11b}$$

In expressions of ψ_1/ψ_2 and $\tilde{\psi}_1/\tilde{\psi}_2$, (ψ_1, ψ_2) and $(\tilde{\psi}_1, \tilde{\psi}_2)$

are homogeneous complex coordinates of parameters λ and μ. Each of equations (11) is a generating line(or generator) of quadric surface $x_1^2+x_2^2+x_3^2+x_4^2=0$.

After substituting x_1, x_2, x_3, and ix_4 with p_x, p_y, p_z, and E respectively in (11), we get

$$-\frac{p_x-ip_y}{p_z+E}=\frac{p_z-E}{p_x+ip_y} = \frac{\psi_1}{\psi_2} \tag{12a}$$

$$-\frac{p_x-ip_y}{p_z-E}=\frac{p_z+E}{p_x+ip_y} = \frac{\tilde{\psi}_1}{\tilde{\psi}_2} \tag{12b}$$

After writing as matrix, they are just Weyl equations:

$$\begin{pmatrix} p_z+E & p_x-ip_y \\ p_x+ip_y & -p_z+E \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{13a}$$

$$\begin{pmatrix} p_z-E & p_x-ip_y \\ p_x+ip_y & -p_z-E \end{pmatrix}\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = 0 \tag{13b}$$

From here, we can see that $\tilde{\psi}=(\tilde{\psi}_1,\tilde{\psi}_2)^T=\sigma_2\psi^*$, where $\psi=(\psi_1,\psi_2)^T$, and superscript T is transposition of matrix. Then we see that Weyl equations are two families of generating lines of constrained quadric $\boldsymbol{p}^2-E^2=0$ in momentum-energy space, and the corresponding two-component spinors (ψ_1,ψ_2) and $(\tilde{\psi}_1,\tilde{\psi}_2)$ are homogeneous coordinates of these two families of generating lines in this quadric surface. So the correspondence between physics and geometry is

Physical object ↔ Geometric object
Weyl equation ↔ line in quadric surface
two-component spinor ↔ homogeneous coordinates of the above line

Now, let us see where the gauge degree of freedom come from. As we see in (11), the ψ_1/ψ_2 will not change as ψ_1 and ψ_2 are transform to $\psi_1\cdot C_0$ and $\psi_2\cdot C_0$, where $C_0=R\cdot\exp(i\alpha)$ may be a complex number or function of coordinates. Because R will be absorb in the normalization constant of wavefunction, there is only one non-physical degree of freedom for this parameter representation, i.e. $\exp(i\alpha)$, that is the gauge transformation

$$\psi\to\psi\exp(i\alpha)$$

of wavefunction of massless fermion.

Based on the above geometric interpretation, we can deduce the internal symmetry of fermions. In $O\mathbb{C}P^3$, line space is Grassmann space $G(2,4)\otimes Z_2=S^2\times S^2$, which has four dimensions, then spinors of fermions have four degrees of freedom, and their compact simple symmetry group is at most $SU(4)$, the maximum real subgroup of transformation group $SL(4,C)$ of $O\mathbb{C}P^3$.

V. GAUGE VECTORS

After finding the geometric interpretation of spinor, we will discuss how to express the gauge vectors by these spinors, because only gauge vector bosons are same fundamental as fermions in standard model.

A quadric surface is the union of two families of generating lines. From equations (11), we can get the expression of this quadric surface by the two parameters λ and μ of generating line, i.e.,

$$x_1:x_2:x_3:x_4=i(1-\mu\lambda):(1+\mu\lambda):i(\mu+\lambda):(\mu-\lambda).$$

After replaced λ,μ by $\psi_1/\psi_2,\psi_1'/\psi_2'$ and x_1,x_2,x_3,x_4 by A_1,A_2,A_3,iA_4, we get

$$\begin{pmatrix} A_1^0 \\ A_2^0 \\ A_3^0 \\ A_4^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1\tilde{\psi}_1' \\ \psi_1\tilde{\psi}_2' \\ \psi_2\tilde{\psi}_1' \\ \psi_2\tilde{\psi}_2' \end{pmatrix}. \tag{14}$$

Because the λ and μ are two variables, so this vector has two degrees of freedom, just as required by gauge vector. We consider this intersection point in quadric surface of three dimensional projective geometry as the geometric correspondence of gauge vector, such as photon.

At the end of section IV we found fermions had only four complex dimensions and their internal symmetry is $SU(4)$ in $O\mathbb{C}P^3$. Because gauge bosons is dyads of two of these spinors, they would correspond with the adjoint representation of fermions. Then the transformation of gauge boson is determined by gauge transformation of fermions. And according to (14), the corresponding gauge vector would transform as below:

$$A_\mu^0\to A_\mu^0\exp(i\alpha)\exp(-i\tilde{\alpha}').$$

VI. LOCAL GAUGE TRANSFORMATION

Now, we will discuss the local transformation of Weyl equation in four dimensional complex vector space $\mathbb{C}^4$ instead of three dimensional oriented projective geometry $O\mathbb{C}P^3$. The constraint $\boldsymbol{p}^2-E^2=0$ is a quadric supersurface in $\mathbb{C}^4$, whose origin is in the point $(0,0,0,0)$. For a general quadric supersurface, whose origin is in $(\boldsymbol{p}_0,E_0)$, this constraint is $(\boldsymbol{p}-\boldsymbol{p}_0)^2-(E-E_0)^2=0$. This is just the local transformation in homogeneous space.

If we combine this local transformation with the gauge one, we will find, the total transformation is

$$\begin{aligned}(\boldsymbol{p},E) &\xrightarrow{\text{localization}} (\boldsymbol{p}-\boldsymbol{p}_0,E-E_0)\\ &\xrightarrow{\text{gauge}} (\boldsymbol{p}-\boldsymbol{p}_0,E-E_0)\exp(i\alpha)\exp(-i\tilde{\alpha}'),\end{aligned}$$

for $(\boldsymbol{p}-\boldsymbol{p}_0,E-E_0)$ also satisfies the equations (14), which transforms as the complex parameters of line,

From above analysis, we will deduce the gauge transformation of gauge vector in Weyl equations. At first, let

$$p_\mu=(\boldsymbol{p},E).$$

we get

$$\sigma_\mu p_\mu\psi=0,$$

where the σ_μ is Pauli matrix and identity 2×2 matrix. After generalizing to the off-origin locality, we get

$$\rightarrow\ \sigma_\mu(p_\mu - A_\mu)\psi = 0.$$

we could insert a identity transformation matrix $I = \exp(-i\alpha)\exp(i\alpha)$ in front of ψ and keep the equation unchanged. Then

$$\rightarrow\ \sigma_\mu(p_\mu - A_\mu)\cdot\exp(-i\alpha)\exp(i\alpha)\psi = 0.$$

Then, we get the global fundamental gauge transformation:

$$\rightarrow\ \sigma_\mu(p_\mu \exp(-i\alpha) - A_\mu \exp(-i\alpha))\exp(i\alpha)\psi = 0.$$

To get the covariant differential, we will focus on the evolvement

$$p_\mu \rightarrow \partial_\mu \rightarrow D_\mu.$$

We take the quantization

$$p_\mu \rightarrow\ \partial_\mu = -i\hbar\partial/\partial x_\mu,$$

and the above expression become

$$\rightarrow\ \partial_\mu \exp(-i\alpha) - eA_\mu \exp(-i\alpha),$$

where e is a coupling constant, such as electronic charge. After expanding the exponential function to second order of α and/or e, we get

$$\rightarrow\ \partial_\mu + \partial_\mu(-i\alpha) + \partial_\mu(-i\alpha)^2/2 - ieA_\mu - eA_\mu(-i\alpha).$$

And deduced to first order of α and/or e:

$$\rightarrow\ \partial_\mu + \partial_\mu(-i\alpha) - ieA_\mu,$$

i.e, the standard covariant differential of gauge principle:

$$\rightarrow\ \partial_\mu - ie(A_\mu + \frac{1}{e}\partial_\mu\alpha)$$

VII. CONCLUSION

Now, we conclude below:

The Weyl equations correspond with equations of generating line of quadric surface in three dimensional oriented projective geometry, and spinor is just the complex parameter coordinate of above line. In these two families of generating lines, every line in one family always intersect with one in another family, then the point of intersection just corresponds with the gauge vector.

The homogeneous parameter coordinate of line allow a non-physical, gauge transformation $\exp(i\alpha)$, which lead to the standard covariant transformation of gauge vector in Weyl equations with interaction. So, the local gauge transformation of fermion and gauge vector is not independent. We also get the global gauge fundamental transformation.

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